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NONABELIAN $N = 2$ SUPERSTRINGS

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ABSTRACT

The Green–Schwarz covariant $N = 2$ superstring action can be consistently deduced as the action of the Wess–Zumino–Witten (WZW) sigma model defined on the direct product of two $N = 1$, $D = 10$ Poincaré supertranslation groups. Generalizing this result, we construct new WZW sigma models on the supergroups with a nonabelian even part and interpret them as models of superstrings moving on the supergroup manifolds. We show that these models are completely integrable and in some special cases possess fermionic κ -symmetry.

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1 INTRODUCTION

An important class of string models is represented by strings moving in a curved background chosen as the product of flat Minkowski space and a nonabelian group manifold [1]. A necessity to consider the models of that kind arises, e.g., while deducing realistic four-dimensional local field theories as a low-energy limit of strings moving in a d -dimensional space-time: the remaining $(d - 4)$ dimensions of the latter are to be compactified in a proper manner (for example, on a group space). Self-consistency of the string theory of that type requires it to be conformally invariant [2]. For ensuring this, the string coordinates valued in the group manifold should be described by the conformally invariant Wess–Zumino–Witten (WZW) sigma-model coupled to the world-sheet gravity. So, the group space string models supply nice examples of a 2D conformal field theory solvable both on classical and quantum levels [1, 3, 4]. It is also worth mentioning that these models provide a way of introducing gauge degrees of freedom on the string [5].

It is tempting to construct the group space analogs of superstrings in the Green–Schwarz (GS) covariant formulation [6], with nonabelian supergroup manifolds as the target ones. These models could offer an appropriate laboratory for getting a further insight into the quantum and algebraic structure of the covariant superstring theory.

In the present paper we propose a general method for formulating such nonabelian superstring models (of the type II) in a consistent way, proceeding from a few primary principles. We confine our study here to the classical case. Full quantum consideration will be given elsewhere.

As distinct from the case of bosonic strings, it is not so easy to define a viable non-abelian generalization of the GS superstring. One of the reasons is that even the flat target superspace possesses a nontrivial geometry in what concerns pure Grassmann dimensions (there are corresponding nonzero torsion components). This gives rise to the existence of non-trivial Wess–Zumino terms which have to be necessarily included in the GS covariant action for reducibility of the latter (in a special gauge) to an action of a free field theory [6]. The reducibility is ensured by local κ -supersymmetry which plays the same role for the GS action as local world-sheet conformal or superconformal symmetries for the actions of bosonic and spinning strings. Any superstring action pretending to be a generalization of the original GS one should either inherit all the remarkable features of the latter or obey instead some self-consistency requirements of another kind (e.g., such could be the property of complete classical and quantum integrability).

What is definitely desirable to maintain is the nice interpretation of GS superstrings as WZW sigma models associated with some superspaces as the target manifolds [7]. However, for the type II superstrings we deal with in the present paper this sigma model interpretation is not straightforward even in the case of a flat target superspace. The point is that the relevant WZ term cannot be immediately related to $10D$ $N = 2$ Poincaré supertranslation algebra which is usually assumed to underly the type II GS superstring (see e.g., [8]). In particular, this term does not possess the automorphism $SO(2)$ symmetry inherent in the above algebra.

In [9] we have argued that for an unambiguous construction of the WZ term the type

II GS action should be defined on the product of two $N = 1$ supertranslation groups rather than on $N = 2$ supertranslation group. Half the original bosonic group parameters is removed owing to the special choice of the target superspace as the coset $G \otimes G/G_+$ where G_+ is an abelian subgroup generated by a linear combination of the two original even translation generators (the product $G \otimes G$ is realized on this coset manifold just as $N = 2$ Poincaré supertranslation group). The WZ term properly constructed out of the Cartan one-forms given on the product $G \otimes G$ is precisely the one entering into the GS action. The breaking of $SO(2)$ symmetry in this term is naturally related to the fact that the supergroup $G \otimes G$ has no such automorphisms.

As a by-product of our interpretation, we have found a zero-curvature representation for the classical covariant GS superstring equations in an arbitrary gauge. The zero-curvature representation superalgebra turned out to be a sum of the two mutually commuting ones, thus reflecting the $G \otimes G$ structure of the GS action. Later on, we extended this construction to an arbitrary supergroup with the commutative even part [10]. The arising generalized GS type sigma models, under a fixed relative strength of the WZ and conventional terms in the action, display complete integrability and, with some further restrictions on the supergroup structure constants, possess local κ -supersymmetry.

The next natural step (and this is precisely what we do here) is to carry out an analogous construction for the essentially nonabelian supergroups G_1, G_2 with non-commutative even parts. The superstring models obtained in this way involve in the bosonic sector the group manifold string models (the WZW sigma models) and thus can be viewed as a genuine superextension of the latter.

In this paper we present the group-theoretical basics of nonabelian superstring models and construct the invariant action for them. We begin in Sec.2 with recalling the main features of our formulation of the ordinary Type II GS superstring. Further, in Sec.3, we extend this to the nonabelian case. As a natural guiding principle, we impose the requirement that the corresponding models are reduced to the GS type models considered in [10] when the underlying nonabelian supergroups contract into those with commutative even parts (we call the latter models abelian superstrings). A novel point is that for ensuring the correct flat superspace limit the supergroups G_1 and G_2 one starts with have to be dual to each other in Cartan's sense [11] ($G_2 = G_1^*$) rather than isomorphic as in the abelian case considered in [9, 10]. From the group-theoretical point of view, the models which we obtain correspond to the nonsymmetric coset $G_1 \otimes G_2/G_+$ where G_+ is a diagonal subgroup in the product of two isomorphic even subgroups of G_1 and $G_2 = G_1^*$. This is a crucial difference of this class of superstring models from, e.g. the model proposed recently by Green [12]. The latter is constructed as a WZW sigma-model for the principal chiral field on a supergroup G which is an extension of the Poincaré supertranslation group by a spinor central charge. From the group-theoretical point of view, it corresponds to the symmetric coset $G_L \otimes G_R/G_{diag}$.

We demonstrate that on the coset $G_1 \otimes G_2/G_+$ it is possible to define a correct WZW action and the latter is reduced just to the standard GS one in the flat superspace limit, when the parameters of the action are adjusted in a proper way. Once again, the models constructed are completely integrable for arbitrary gauge fixing. What concerns local κ -

supersymmetry, it is present in its standard form only under a certain additional condition on the structure constants, like in the abelian case [10].

In further publications we are planning to investigate the Hamiltonian structure of our nonabelian models (along the line of Ref. [13])

2 $N = 2$ GREEN–SCHWARZ SUPERSTRING AS A WZW SIGMA MODEL ON THE PRODUCT OF TWO $N = 1$ SUPERTRANSLATION GROUPS

To make more clear the motivations for considering the superstring-like models constructed in the next Section, it is instructive to start with recalling the basic points of our interpretation of the $N = 2$ GS superstring [9].

2.1 Underlying concepts

The major statement of [9] is that for achieving the complete analogy between 2D WZW sigma models and $N = 2$ GS superstring (both of IIA and IIB types) the latter has to be constructed on the product of two $N = 1$ supertranslation groups rather than on the $N = 2$ supergroup as it has been originally proposed in [7, 8]. This allows an algorithmic construction of the $N = 2$ superstring WZ term out of the Cartan 1-forms defined on the above product. $N = 2$ supersymmetry of the action is ensured due to a special choice of the target manifold as a proper coset of this product of supergroups.

In more detail, the basic concepts of our construction are as follows:

A. One starts with the product of two isomorphic $N = 1$, $D = 10$ supertranslation groups $G_1 \otimes G_2$ generated, respectively, by the generators (P_μ^1, Q_α^1) and $(P_\mu^2, Q_\alpha^2)^2$

$$\begin{aligned} \{Q_\alpha^1, Q_\beta^1\} &= -\Gamma_{\alpha\beta}^\mu P_\mu^1, [P_\mu^1, Q_\alpha^1] = [P_\mu^1, P_\nu^1] = 0, \\ \{Q_\alpha^2, Q_\beta^2\} &= \Gamma_{\alpha\beta}^\mu P_\mu^2, [P_\mu^2, Q_\alpha^2] = [P_\mu^2, P_\nu^2] = 0 \end{aligned} \quad (2.1)$$

(different signs in the r.h.s. of anticommutators of spinor charges are chosen for the reason to be clear later). The odd generators are assumed to be Majorana–Weyl spinors with 16 independent components so $\Gamma_{\alpha\beta}^\mu$ in (2.1) should actually include the corresponding projectors. It will be more convenient for us to impose the Weyl condition on the Grassman coordinates associated with Q_α^1, Q_α^2 rather than to explicitly incorporate it into the structure relations (2.1). One may choose chiralities of Q_α^1, Q_α^2 in an arbitrary way: the case of the same chiralities corresponds to the IIB type GS superstring while opposite chiralities lead to the IIA type. The $D = 10$ Lorentz group is treated as an outer automorphism group of (2.1) acting on the vector μ, ν, λ, \dots and spinor $\alpha, \beta, \gamma, \dots$ indices. It should be

²We use the standard $D = 10$ conventions $\Gamma_{\alpha\beta}^\mu = (\gamma^\mu)_\alpha^\delta C_{\delta\beta} = \eta^{\mu\nu} \Gamma_{\nu, \alpha\beta}$ where γ^μ are 32×32 $D = 10$ Dirac γ -matrices, $C_{\alpha\beta} = -C_{\beta\alpha}$ is the charge conjugate matrix and $\eta^{\mu\nu} = \text{diag}(1, -1, -1, \dots, -1)$.

emphasized that (2.1) possesses neither $SO(2)$ nor $SO(1,1)$ groups of automorphisms; this becomes clear after rewriting (2.1) as

$$\begin{aligned} \{Q_\alpha^i, Q_\beta^j\} &= -\Gamma_{\alpha\beta}^\mu (P_\mu^+ \sigma_3^{ij} + P_\mu^- \delta^{ij}) , \\ [P_\mu^\pm, Q_\alpha^i] &= 0, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P_\mu^\pm = \frac{1}{2}(P_\mu^1 \pm P_\mu^2) \end{aligned} \quad (2.2)$$

For future use, we note that the reductions $P_\mu^+ = 0$ or $P_\mu^- = 0$ in (2.2) lead, respectively, to the two kinds of $N = 2$ $D = 10$ superslgebras, the standard one with the $SO(2)$ automorphism group and the “noncompact” one with the $SO(1,1)$ automorphisms. The presence of the additional vector generator P_μ^+ in (2.2) will turn out crucial for an algorithmic construction of the WZ term: the latter will be shown to essentially involve the Cartan form associated with this generator.

B. The target space of our sigma model is identified with the coset $G_1 \otimes G_2 / G^+$ where G^+ is the abelian translation subgroup generated by P_μ^+ .

This space actually coincides with ordinary $N = 2$ $D = 10$ superspace. Indeed, let us parametrize the elements of G_1, G_2 as

$$U_j = \exp \left\{ i \left(\frac{1}{2} x^{j\mu} P_\mu^j + \theta^{j\alpha} Q_\alpha^j \right) \right\}, \quad j = 1, 2 \quad (2.3)$$

The left action of G_1, G_2 on these elements induces the supersymmetry transformations in the two $N = 1$ $D = 10$ superspaces $\{x^{1\mu}, \theta^{1\alpha}\}$ and $\{x^{2\mu}, \theta^{2\alpha}\}$

$$\begin{aligned} x^{1\mu'} &= x^{1\mu} + i\varepsilon^{1\alpha} \Gamma_{\alpha\beta}^\mu \theta^{1\beta}, \quad \theta^{j\beta'} = \theta^{j\beta} + \varepsilon^{j\beta}, \\ x^{2\mu'} &= x^{2\mu} - i\varepsilon^{2\alpha} \Gamma_{\alpha\beta}^\mu \theta^{2\beta}. \end{aligned} \quad (2.4)$$

Considering the product $U_1 U_2$ and factoring out the exponent with the G_+ generator P_μ^+ one represents elements of $G_1 \otimes G_2 / G_+$ as

$$\{G_1 \otimes G_2 / G_+\} = \exp \left\{ i x^\mu P_\mu^- + i \sum_j \theta^{j\alpha} Q_\alpha^j \right\} \quad (2.5)$$

with

$$x^\mu = x^{1\mu} - x^{2\mu}.$$

From (2.4) it follows that $G_1 \otimes G_2$ acts on the set $\{x^\mu, \theta^{1\alpha}, \theta^{2\alpha}\}$ in precisely the same way as $N = 2$ $D = 10$ supergroup

$$x^{\mu'} = x^\mu + \sum_j \varepsilon^{j\alpha} \Gamma_{\alpha\beta}^\mu \theta^{j\beta}, \quad \theta^{j\alpha'} = \theta^{j\alpha} + \varepsilon^{j\alpha} \quad (2.6)$$

Another way to realize that $G_1 \otimes G_2$ is undistinguishable from the $N = 2$ Poincaré supergroup when applied to $\{x^\mu, \theta^{j\alpha}\}$ is to take notice of the fact that P_μ^+ is zero on this coordinate set and so (2.2) actually reduces on the latter to $N = 2$ supertranslation algebra. The difference of (2.2) from $N = 2$ superalgebra actually manifests itself only in

the existence of an inhomogeneously transforming Cartan's form on $\{x^\mu, \theta^{j\alpha}\}$ associated with P_μ^+ (this is a P_μ^+ connection).

The construction of a 2D nonlinear sigma model based on these two assumptions follows the standard routine. The building blocks are the left-invariant one-forms on the supergroups G_1 and G_2

$$\begin{aligned} u_j^{-1}(\xi) \partial_a U_j(\xi) &= \omega_a^{j\mu} P_\mu^j + \omega_a^{j\alpha} Q_\alpha^j = \\ &= \left[\frac{i}{2} (\partial_a x^{j\mu} - i(-1)^j \partial_a \theta^j \Gamma^\mu \theta^j) \right] P_\mu^j + [i \partial_a \theta^{j\alpha}] Q_\alpha^j, \end{aligned} \quad (2.7)$$

where $x^{j\mu}$ and $\theta^{j\alpha}$ are treated as the scalar fields on the world sheet $\xi = (\xi^0, \xi^1)$. The Cartan-forms describing the coset $G_1 \otimes G_2 / G_+$ (2.5) are composed of these objects in an evident way

$$\begin{aligned} \omega_a^\mu &= \omega_a^{1\mu} - \omega_a^{2\mu} = \frac{i}{2} \left(\partial_a x^\mu + i \sum_{j=1}^2 \partial_a \theta^j \Gamma^\mu \theta^j \right), \\ \omega_a^{j\alpha} &= i \partial_a \theta^{j\alpha}, \\ \Omega_a^\mu &= -\frac{1}{2} \sum_{j,k} \partial_a \theta^j \Gamma^\mu (\sigma_3)^{jk} \theta^k \end{aligned} \quad (2.8)$$

and are introduced by the generic nonlinear-realization formula ($x^{1\mu} + x^{2\mu} = 0$)

$$\omega_a \equiv U_2^{-1}(\xi) U_1^{-1}(\xi) \partial_a [U_1(\xi) U_2(\xi)] = \omega_a^\mu P_\mu^- + \sum_{j=1}^2 \omega_a^{j\alpha} Q_\alpha^j + \Omega_a^\mu P_\mu^+ \quad (2.9)$$

Here the Cartan forms $\omega_a^\mu d\xi^a$, $\omega_a^{j\alpha} d\xi^a$ associated with the coset generators P_μ^- , Q_α^j are covariant differentials of the coset fields $x^\mu(\xi)$, $\theta^{j\alpha}(\xi)$. The one-form $\Omega_a^\mu d\xi^a$ is the connection on the stability subgroup G_+ .

It is convenient to single out the coset $G_1 \otimes G_2 / G_+$ from the product $G_1 \otimes G_2$ by imposing invariance under the right gauge G_+ transformations

$$\begin{aligned} U_1(\xi) &\rightarrow U_1(\xi) \exp\{y^\mu(\xi) P_\mu^1\}, \\ U_2(\xi) &\rightarrow U_2(\xi) \exp\{y^\mu(\xi) P_\mu^2\} \\ \omega_a^{j\mu} &\rightarrow \omega_a^{j\mu} + \partial_a y^\mu(\xi), \quad \omega_a^{j\alpha} \rightarrow \omega_a^{j\alpha} \end{aligned} \quad (2.10)$$

Any action invariant under these gauge transformations actually involves only the coset space coordinates $\{x^\mu, \theta^{j\alpha}\}$. The Cartan forms (2.8), (2.9) correspond to a particular fixing of this gauge freedom, so as

$$x^{1\mu} = -x^{2\mu} = \frac{1}{2} x^\mu$$

The useful object is

$$U(\xi) = U_1(\xi) U_2^{-1}(\xi) =$$

$$= \exp \left\{ i \frac{1}{2} x^{1\mu} P_\mu + i \theta^{1\alpha} Q_\alpha \right\} \exp \left\{ -i \frac{1}{2} x^{2\mu} P_\mu - \theta^{2\alpha} Q_\alpha \right\} \quad (2.11)$$

where we have substituted $P_\mu^1 = P_\mu^2 = P_\mu$, $Q^{1\mu} = iQ^{2\alpha} = Q^\alpha$ (formally, $-iQ^\alpha$, P^μ satisfy the same superalgebra as $Q^{2\alpha}$, P_μ^2). This supergroup element is invariant under the gauge transformations (2.10) and so lives on the coset $G_1 \otimes G_2/G_+$. The supergroups G_1 and G_2 act as the left and right multiplications of $U(\xi)$

$$U'(\xi) = g_1 U(\xi) g_2^{-1}, \quad g_1 \in G_1, \quad g_2 \in G_2 \quad (2.12)$$

This matrix field is an analog of the standard principal chiral fields, the only difference is that the latter describes symmetric cosets of the type $G_L \otimes G_R/G_{diag}$ ($G_L \sim G_R$ can be arbitrary groups or supergroups) and is invariant under right gauge G_{diag} transformations while in the case at hand we deal with the nonsymmetric coset. Correspondingly, from $U(\xi)$ one may construct the “left” and “right” linearly transforming currents

$$\begin{aligned} J_a^L(\xi) &= U \partial_a U^{-1} = -U_1 [\omega_a^\mu P_\mu + (\omega_a^{1\alpha} + i\omega_a^{2\alpha}) Q_\alpha] U_1^{-1}, \\ J_a^R(\xi) &= U^{-1} \partial_a U = U_2 [\omega_a^\mu P_\mu + (\omega_a^{1\alpha} + i\omega_a^{2\alpha}) Q_\alpha] U_2^{-1}, \\ J_a^- &= J_a^{L-} = J_a^{R-} = U_1 \omega_a^\mu P_\mu U_1^{-1} = U_2 \omega_a^\mu P_\mu U_2^{-1} = \omega_a^\mu P_\mu, \\ J_a^{L'} &= g_1 J_a^L g_1^{-1} \quad J_a^{R'} = g_2 J_a^R g_2^{-1}. \end{aligned} \quad (2.13a)$$

However, in the present case, besides J_a^L , J_a^R , one may define two more objects

$$J_a^{Lj} = U_1 \omega_a^{j\alpha} Q_\alpha U_1^{-1}, \quad J_a^{Rj} = U_2 \omega_a^{j\alpha} Q_\alpha U_2^{-1} \quad (2.13b)$$

transforming in the same way as J_a^L , J_a^R . This is because our coset space is nonsymmetric and, as a consequence, there exist on it three independent homogeneously transforming Cartan's forms $\omega_a^\mu d\xi^a$, $\omega_a^{1\alpha} d\xi^a$, $\omega_a^{2\alpha} d\xi^a$ and each can be used to construct a current-like quantity.

2.2 The $N = 2$ GS action as a WZW action

The generic form of the action of WZW sigma associated with the coset $G_1 \otimes G_2/G_+$ is as follows [9, 10]

$$A = \int_{\partial V} d^2 \xi \sqrt{-g} g^{ab} \langle \omega_a \omega_b \rangle_I + \frac{1}{2} \int_V d^3 \xi \varepsilon^{ABC} \langle \omega_A [\omega_B, \omega_C] \rangle_{II} \quad (2.14)$$

Here $\omega_a d\xi^a$, $\omega_A d\xi^A$ ($A = 1, 2, 3$) are Cartan's forms defined by Eq.(2.9), ∂V is a two-dimensional boundary of the three-dimensional region V and $g^{ab}(\xi)$ is a metric on ∂V . The symbols $\langle \dots \rangle_{I,II}$ stand for a kind of the metric on superalgebras (2.1). As is well-known, it is impossible to define the cyclic operation Str for the supertranslation algebras (2.1) and, respectively, an invariant non-degenerate metric. So, $\langle \dots \rangle_{I,II}$ in (2.14) should be regarded rather as some general bilinear forms of the Cartan forms $\omega_a^\mu, \omega_a^{1\alpha}, \omega_a^{2\alpha}, \Omega_a^\mu$

with the coefficients unspecified for the moment. These coefficients are almost uniquely (up to two parameters) fixed by the following four natural principles

- 1) $D = 10$ Lorentz invariance and rigid G_1, G_2 supersymmetry;
- 2) Gauge (local) right G_+ invariance;
- 3) The closeness of the WZ three-form

$$\Omega_3 = \langle \omega \wedge \omega \wedge \omega \rangle_{II}, \quad \delta \Omega_3 = d\Omega_2$$

where Ω_2 is a two-form.

- 4) The absence of the fermionic kinetic terms having the second order in $\partial/\partial\xi^0$.

These principles specify the “metrics” $\langle \dots \rangle_{I,II}$ up to two constants ℓ_I, ℓ_{II}

$$\langle P_\mu^- P_\nu^- \rangle_I = \ell_I \eta_{\mu\nu}, \quad \langle P_\mu^-, P_\nu^+ \rangle_{II} = \ell_{II} \eta_{\mu\nu}, \quad (2.15)$$

all the remaining “averages” of bilinear forms in the generators being zero. Correspondingly, the action takes the form

$$\begin{aligned} A = & \ell_I \int_{\partial V} d^2 \xi \sqrt{-g} g^{ab} \omega_a^\mu \omega_{b\mu} + \\ & + \frac{1}{2} \ell_{II} \int_V d^3 \xi \varepsilon^{ABC} \omega_A^\mu \omega_B^{j\alpha} \Gamma_{\mu, \alpha\beta}(\sigma_3)^{ij} \omega_C^{i\beta}, \end{aligned} \quad (2.16)$$

or

$$\begin{aligned} A = & \ell_I \int_{\partial V} d^2 \xi \sqrt{-g} g^{ab} \omega_a^\mu \omega_{b\mu} + \\ & + \ell_{II} \int d^3 \xi \varepsilon^{ABC} \omega_A^\mu \partial_B \Omega_{C\mu} \end{aligned} \quad (2.16')$$

where in achieving two equivalent forms of the WZ term we have made use of the Maurer–Cartan equations for ω_A to rewrite the product of the spinor 1-forms as $\partial_{[B} \Omega_{C]}^\mu$. Note a natural appearance of the matrix $(\sigma_3)^{ij}$ in the WZ term (2.16) as the structure constants standing before the generator P_n^+ in the superalgebra (2.2) we have started with. This matrix has come out as the result of the evaluation of the commutator $[\omega_B, \omega_C]$ in (2.14). At the same time, in the standard view on $N = 2$ GS superstring, it is introduced to some extent, “by hand” [7, 8] because it is not present in $N = 2$ $D = 10$ supertranslation algebra. The breaking of $SO(2)$ automorphism symmetry in the WZ term gets now an explanation as connected with the fact that the superalgebra (2.2) possesses no internal automorphisms at all (neither $SO(2)$ nor $SO(1, 1)$). A subtle point is, of course, a specific choice of the “metrics” $\langle \dots \rangle_{I,II}$ in (2.15) which cannot be immediately related to the inner structure of superalgebra (2.2). An analogy with the WZ terms of ordinary sigma models becomes even more direct in the case of non-abelian superstrings constructed along similar lines in the next section. For nonabelian analogs of superalgebra (2.2) one is able to define the cyclic operation Str and, respectively, a nondegenerate metric, so the symbols $\langle \dots \rangle_{I,II}$ become well defined in this case. Upon contraction to the flat superalgebra (2.2), one finds that the limiting action involves just the “metrics” (2.15). Thus the choice (2.15) can in fact be justified by resorting to the nondegenerate case of nonabelian superstrings.

We wish to point out that the second translation generator P_μ^+ manifests itself in the present context only in giving rise to an additional Cartan form Ω^μ which underlies the $N = 2$ GS WZ term. In this aspect, P_μ^+ resembles the central charge operators in the geometric formulations of $N \geq 2$ 4D supergravities. These operators serve to render a geometric meaning to some members of the relevant gauge multiplets (e.g. to the graviphoton in $N = 2$ Einstein supergravity), however, in their own right, do not generate any symmetry of the latter. Note that the nonabelian analogs of P_μ^+ possess a nontrivial action on the target superspace coordinates (see next Section).

Let us come back to the discussion of (2.16). Rewriting the WZ term as an integral over ∂V with the help of the important identity for Dirac $D = 10$ γ matrices

$$\Gamma_{\alpha,\beta}^\mu \Gamma_{\mu,\gamma\delta} + (\text{cyclic } (\alpha, \beta, \gamma)) = 0 \quad (2.17)$$

we obtain the GS covariant “preaction” containing two free parameters ($\varepsilon^{01} = -\varepsilon^{10} = 1$)

$$\begin{aligned} A = & -\frac{\ell_I}{4} \int_{\partial V} d^2\xi \left\{ \sqrt{-g} g^{ab} \left(\partial_a x^\mu + i \sum_{j=1}^2 \partial_a \theta^j \Gamma^\mu \theta^j \right) \left(\partial_b x_\mu + i \sum_{j=1}^2 \partial_b \theta^j \Gamma_\mu \theta^j \right) - \right. \\ & \left. - \frac{\ell_{II}}{\ell_I} \varepsilon^{ab} \left(i \partial_a x^\mu - \frac{1}{2} \sum_{j=1}^2 \partial_a \theta^j \Gamma^\mu \theta^j \right) \left(\sum_{k,j=1}^2 \partial_b \theta^k \sigma_3^{kj} \Gamma_\mu \theta^j \right) \right\} \end{aligned} \quad (2.18)$$

The genuine GS action possessing local κ -supersymmetry arises at $\ell_{II}/\ell_I = 2$ [6]. The κ -symmetry transformations are given by

$$\begin{aligned} \tilde{\omega}^{1\mu} + \tilde{\omega}^{2\mu} &= 0 \\ \tilde{\omega}^{1\alpha} &= P_+^{ab} (\omega_{b\mu}^1 - \omega_{b\mu}^2) \tilde{\Gamma}^{\mu,\alpha\beta} \kappa_{a\beta}^1(\xi) \\ \tilde{\omega}^{2\alpha} &= P_-^{ab} (\omega_{b\mu}^1 - \omega_{b\mu}^2) \tilde{\Gamma}^{\mu,\alpha\beta} \kappa_{a\beta}^2(\xi) \\ \delta(\sqrt{-g} g^{ab}) &= -2(P_+^{da} P_+^{cb} \omega_d^{1\alpha} \kappa_{c\alpha}^1 + P_-^{da} P_-^{cb} \omega_d^{2\alpha} \kappa_{c\alpha}^2) \end{aligned} \quad (2.19)$$

where $\kappa_{\alpha\beta}^j$ are transformation parameters, $\tilde{\Gamma}^\mu = C^{-1} \gamma^\mu$, P_\pm^{ab} are 2D light-cone projectors and we have introduced the left-invariant variations

$$(U_j)^{-1} \delta U_j = \tilde{\omega}^{j\mu} P_\mu^j + \tilde{\omega}^{j\alpha} Q_\alpha^j$$

2.3 A zero-curvature representation for the GS superstring equations

In terms of Cartan’s forms the GS superstring equations of motion are written as

$$\partial_a [P_+^{ab} \omega_b^\mu - \frac{\ell_{II}}{\ell_I} \varepsilon^{ab} \omega_b^{1\mu}] = 0, \quad (2.20)$$

$$\partial_a [P_-^{ab} \omega_b^\mu - \frac{\ell_{II}}{\ell_I} \varepsilon^{ab} \omega_b^{2\mu}] = 0,$$

$$\begin{aligned}
P_+^{ab}\omega_{b\mu}\Gamma_{\alpha\beta}^\mu\omega_a^{1\beta} &= 0 , \\
P_-^{ab}\omega_{b\mu}\Gamma_{\alpha\beta}^\mu\omega_a^{2\beta} &= 0
\end{aligned} \tag{2.21}$$

where $P_\pm^{ab} = \sqrt{-g}g^{ab} \pm \frac{\ell_{II}}{2\ell_I}\varepsilon^{ab}$. We have to add to (2.20) and (2.21) the Maurer–Cartan equations and the equation

$$\omega_a^\mu\omega_{b\mu} - \frac{1}{2}g_{ab}g^{cd}\omega_c^\mu\omega_{d\mu} = 0 \tag{2.22}$$

which is obtained by varying g^{ab} in (2.16). The set of the basic GS superstring Eqs.(2.20), (2.21) admits a reformulation as the conservation laws for the appropriate G_1 and G_2 currents (with the Maurer–Cartan equations taken into account)

$$\begin{aligned}
\partial_a[P_+^{ab}\tilde{J}_b^1 - \frac{\ell_{II}}{\ell_I}\varepsilon^{ab}\tilde{J}_b^{L1}] &= 0 , \\
\partial_a[P_-^{ab}\tilde{J}_b^2 - \frac{\ell_{II}}{\ell_I}\varepsilon^{ab}\tilde{J}_b^{R2}] &= 0 ,
\end{aligned} \tag{2.23}$$

where the quantities in the square brackets are

$$\begin{aligned}
\tilde{J}_b^j &= \omega_b^\mu P_\mu + \omega_b^\mu(\theta^j\gamma_\mu Q) \\
\tilde{J}_b^{L1} &= \omega_b^{1\mu}P_\mu + \frac{1}{3}(\theta^1\Gamma_\mu\partial_b\theta^1)(\theta^1\gamma^\mu Q) \\
\tilde{J}_b^{R2} &= \omega_b^{2\mu}P_\mu - \frac{1}{3}(\theta^2\Gamma_\mu\partial_b\theta^2)(\theta^2\gamma^\mu Q)
\end{aligned}$$

Surprisingly, the GS Eqs.(2.20), (2.21) admit a zero curvature representation, precisely at the same value of $\ell_{II}/\ell_I = 2$ which is selected by κ -invariance. At this specific value of ℓ_{II}/ℓ_I one may rewrite these equations as the integrability conditions

$$[L_a^1, L_b^1] = [L_a^2, L_b^2] = 0 , \tag{2.24}$$

$$\begin{aligned}
L_a^1 &= \partial_a - 2\lambda^2\varepsilon_{ab}P_+^{bc}\omega_c^\mu R_\mu^1 + 2\lambda i\omega_a^{1\alpha}S_\alpha^1 , \\
L_a^2 &= \partial_a + 2\lambda^2\varepsilon_{ab}P_-^{bc}\omega_c^\mu R_\mu^2 + 2\lambda i\omega_a^{2\alpha}S_\alpha^2 ,
\end{aligned} \tag{2.25}$$

where λ is a spectral parameter and (R_μ^j, S_α^j) constitute two mutually commuting super-algebras [9]

$$[R_\mu^j, R_\nu^j] = R_{[\mu\nu]}^j , \tag{2.26a}$$

$$[R_\mu^j, S_\alpha^j] = \Gamma_{\mu,\alpha\beta}\tilde{S}^{j\beta} , \tag{2.26b}$$

$$\{S_\alpha^j, S_\beta^j\} = -\Gamma_{\alpha\beta}^\mu R_\mu^j . \tag{2.26c}$$

Note that the generator $\tilde{S}^{j\beta}$ cannot be put proportional to $S^{j\beta}$ without contradiction between Eqs. (2.26b), (2.26c) and the assumption that $R_{[\mu\nu]}^j \neq t_{\mu\nu}^\lambda R_\lambda^j$. Actually only the relations (2.26b) and (2.26c) are of immediate relevance for deducing Eqs.(2.20), (2.21)

from (2.24), (2.25). The form of the relation (2.26a) is not fixed (nonzero $R_{[\mu\nu]}^j$ seem to be of need only while constructing an infinite number of the conserved currents associated with the Lax pair (2.24)). Also, the newly introduced generator $\tilde{S}^{j\beta}$ is not obliged to commute with the old ones and with itself. So, the complete zero-curvature representation superalgebra can be in principle infinite-dimensional. The minimal possibility which yet allows one to establish the equivalence of Eqs.(2.20), (2.21) with (2.24) and (2.25) is to put $R_{[\mu\nu]}^j = 0$, $[\tilde{S}, R_\mu] = \{\tilde{S}, \tilde{S}\} = 0$ in (2.24), (2.26). The corresponding superalgebra is known now as Green's superalgebra [12]. Thus, the $N = 2$ GS superstring equations admit a minimal zero-curvature representation on the two mutually commuting Green's superalgebras.

It is interesting to mention that in the language of the zero-curvature representation the local κ -symmetry of the GS equations reveals itself as a kind of gauge transformations of the operators L_a^1, L_a^2 preserving the representation (2.24)

$$L_a^j \rightarrow G_j(\xi) L_a^j G_j^{-1}(\xi) \quad (2.27)$$

where, in the infinitesimal form,

$$\begin{aligned} G_1(\xi) &= 1 - 2i\lambda\tilde{\omega}^{1\alpha} S_\alpha^1 \\ G_2(\xi) &= 1 - 2i\lambda\tilde{\omega}^{2\alpha} S_\alpha^2 \end{aligned} \quad (2.28)$$

and variations $\tilde{\omega}^{j\alpha}$ were defined in Eqs.(2.19) (when checking invariance of (2.20), (2.21) under (2.27), (2.28), one has to take into account the equations of motion).

2.4 Summary and comments

To summarize, the sigma-model interpretation of the covariant action of $N = 2$ GS superstring is most naturally achieved provided one starts with the product of two $N = 1$ D=10 Poincaré supertranslation groups $G_1 \otimes G_2$ and constructs the WZW sigma model on the coset space $G_1 \otimes G_2 / G_+$ where G_+ is the diagonal translation subgroup.

It is natural to ask how the present construction could be extended to include the supergroups different from the $N = 1$ supertranslation ones, in particular those with a nonabelian even part, and how all this matches with $N = 2$ GS superstring moving in an arbitrary $N = 2$ $D = 10$ supergravity background. The first question was partly addressed in our paper [10] where we considered the superstring-like sigma models associated with the more general superalgebras of the type (2.10) which were obtained by contraction from arbitrary Lie superalgebras. In this way we have arrived at the actions of the kind (2.17) in which, however, the structure constants $\Gamma_{\alpha\beta}^\mu$ do not in general coincide with $\gamma^\mu C$. These models still reveal the property of classical integrability, but the local κ -symmetry is present only under some severe restrictions on the constants $\Gamma_{\alpha\beta}^\mu$.

In [9, 10] we have regarded G_1 and G_2 to be “abelian”, i.e. possessing the superalgebras of the type (2.1), in which the even generators commute with each other and with the odd ones. In the next section we shall study the most general situation, with the above restrictions on G_1, G_2 removed. The construction outlined here will be shown to

work equally well also in this general case. It yields a new wide class of self-consistent WZW sigma models admitting an interpretation in terms of the superstring moving in a nonabelian supergroup background.

3 THE GREEN–SCHWARZ TYPE SUPERSTRING ON THE LIE SUPERGROUP COSET SPACE

Our aim here is to generalize the construction expounded in Sec.2 to the case of an arbitrary Lie supergroup with nondegenerate metric. The resulting sigma model will be interpreted as a theory of superstring moving on a curved coset supermanifold.

3.1 Supergroup preliminaries

Let $G_1(d|D)$ be a Lie supergroup with the even and odd generators $R_\mu^1(\mu, \nu, \lambda, \dots = 1, 2, \dots, d)$ and $S_\alpha^1(\alpha, \beta, \gamma, \dots = 1, 2, \dots, D)$ which obey the following (anti)commutation relations

$$[R_\mu^1, R_\nu^1] = t_{\mu\nu}^\lambda R_\lambda^1, \{S_\alpha^1, S_\beta^1\} = -\Gamma_{\alpha\beta}^\mu R_\mu^1, [R_\mu^1, S_\alpha^1] = C_{\mu\alpha}^\beta S_\beta^1, \quad (3.1)$$

$\Gamma_{\alpha\beta}^\mu, C_{\mu\alpha}^\beta, t_{\mu\nu}^\lambda$ being the structure constants. The only restriction we impose on (3.1) is the existence of a nondegenerate metric

$$\eta_{\mu\nu} = \eta_{\nu\mu} = -Str(R_\mu^1 R_\nu^1), \quad X_{\alpha\beta} = -X_{\beta\alpha} = Str(S_\alpha^1 S_\beta^1), \quad Str(R_\mu^1 S_\alpha^1) = 0$$

Then the evident identity

$$Str(\{S_\alpha^1, S_\beta^1\} R_\mu^1) = Str(S_\alpha^1 [S_\beta^1, R_\mu^1])$$

yields [10]

$$\Gamma_{\alpha\beta}^\mu \eta_{\mu\nu} = C_{\mu\beta}^\gamma X_{\gamma\alpha}$$

or

$$C_{\mu\beta}^\alpha = \eta_{\mu\nu} \Gamma_{\beta\gamma}^\nu X^{\gamma\alpha} \quad (X_{\alpha\gamma} X^{\gamma\beta} = \delta_\alpha^\beta)$$

From the identity

$$Str(\{S_\alpha^1, S_\beta^1\} \{S_\gamma^1, S_\delta^1\} + (\text{cyclic } \alpha, \beta, \gamma)) = 0$$

one deduces the important relation (compare with Eq.(2.17))

$$[\Gamma_{\alpha\beta}^\mu \Gamma_{\gamma\delta}^\nu + (\text{cyclic } \alpha, \beta, \gamma)] \eta_{\mu\nu} = 0 \quad (3.2)$$

From the Jacobi identities there also follow certain quadratic relations between the structure constants. These can be easily derived, so we do not give them explicitly (see, e.g. [10]). The described class of superalgebras is very wide: it includes all the semisimple Lie superalgebras and also some non-semisimple ones, e.g. those of the type considered by

Green [12]. Given the superalgebra (3.1) one may construct a new superalgebra by the formal substitution $S_\alpha^1 \rightarrow iS_\alpha^2$ in (3.1) ($R_\mu^1 \rightarrow R_\mu^2$)

$$[R_\mu^2 R_\nu^2] = t_{\mu\nu}^\lambda R_\lambda^2, \{S_\alpha^2, S_\beta^2\} = +\Gamma_{\alpha\beta}^\mu R_\mu^2, [R_\mu^2, S_\alpha^2] = C_{\mu\alpha}^\beta S_\beta^2 \quad (3.3)$$

This superalgebra is called dual in Cartan's sense to (3.1) [11]³. Both (3.1) and (3.3) may be regarded as different real forms of the superalgebra $g^c(d|D)$ of the complex supergroup $G^c(d|D)$ generated by the double set of generators (R_μ^1, iR_μ^1) , $(S_\alpha^1, iS_\alpha^1)$. It is easy to check that the pairs (R_μ^1, S_α^1) and (R_μ^1, iS_α^1) constitute in g^c two subsuperalgebras which are isomorphic to (3.1) and (3.3) and have as a crossover the even subalgebra with the generators R_μ^1 . In what follows we shall regard (3.1) and (3.3) as two independent mutually commuting Lie superalgebras.

Generally, (3.1) and (3.3) are not isomorphic to each other. They become isomorphic in the degenerate limit $C_{\mu\alpha}^\beta = t_{\mu\nu}^\lambda = 0$ corresponding to the case of flat superspace treated in Sec.2 (in this case, (3.3) takes the form (3.1) after the redefinition $R_\mu^2 \rightarrow -R_\mu^2$)⁴

The basic motivation for considering the dual superalgebra (3.3) in parallel with (3.1) comes from the desire to have a correct generalization of the flat superspace construction of Sec.1 to curved case. As it will become clear later, this is possible only provided one construct the WZW sigma model on the product $G = G_1 \otimes G_2$ where the supergroups G_1 and G_2 correspond to dual superalgebras (3.1) and (3.3) and so are dual to each other. Only under this choice, in the flat superspace limit one recovers the standard GS superstring type model (see Subsec.2.4). From the mathematical point of view, $G = G_1 \otimes G_2$ is distinguished in that it is self-dual in Cartan's sense.

3.2 Nonlinear realization of $G_1 \otimes G_2$

In constructing the $G_1 \otimes G_2$ WZW sigma model we shall closely follow the consideration in Sec.2. The elements of G_1 and G_2 are parametrized as

$$U_1 = \exp \left\{ \frac{i}{2} x^{1\mu} R_\mu^1 + i\theta^{1\alpha} S_\alpha^1 \right\}$$

$$U_2 = \exp \left\{ \frac{i}{2} x^{2\mu} R_\mu^2 + i\theta^{2\alpha} S_\alpha^2 \right\} = \exp \left\{ \frac{i}{2} x^{2\mu} R_\mu^2 + \theta^{2\alpha} S_\alpha \right\} \quad (3.4)$$

$$S_\alpha = iS_\alpha^2, \{S_\alpha, S_\beta\} = -\Gamma_{\alpha\beta}^\mu R_\mu^2, \{S_\alpha^j, \theta^{k\beta}\} = 0 \quad (3.5)$$

with $\{x^{1\mu}, x^{2\mu}\}$ and $\{\theta^{1\alpha}, \theta^{2\alpha}\}$ being even and odd real supergroup parameters. it is seen that U_1 and U_2 can be viewed as two different restrictions of a generic element of the complex supergroup $G^c(d|D)$

$$U^c = \exp \left\{ \frac{i}{2} (x^\mu + iy^\mu) R_\mu + i(\theta^\alpha + i\eta^\alpha) S_\alpha \right\}$$

³Dual (super)algebra can be defined for any (super)algebra admitting a Z_2 grading. For instance, dual to $SU(2)$ is $SU(1,1)$.

⁴The superalgebras (3.1) and (3.3) can also be isomorphic if $t_{\mu\nu}^\lambda = 0$, $C_{\mu\alpha}^\beta \neq 0$. An example of that sort is given by Green's superalgebra [12].

$$U^c \rightarrow U_1 \text{ if } y^\mu = 0, \eta^\alpha = 0$$

$$U^c \rightarrow U_2 \text{ if } y^\mu = 0, \theta^\alpha = 0$$

Further, U_1 and U_2 are assumed to be defined on the world sheet (ξ^0, ξ^1)

$$U_i = U_i(\xi^0, \xi^1)$$

To have a kind of $N = 2$ superspace as the sigma model target space, we should identify the latter with the homogeneous space $G_1 \otimes G_2/G_+$ where G_+ is the diagonal even subgroup generated by

$$R_\mu^+ = \frac{1}{2}(R_\mu^1 + R_\mu^2) \quad (3.6)$$

The elements of the coset $G_1 \otimes G_2/G_+$ are represented by the group orbit

$$U_1 U_2 \exp\{i y^\mu(\xi) R_\mu^+\}$$

where $y^\mu(\xi)$ are arbitrary world-sheet fields. So, the functionals defined on the coset space $G_1 \otimes G_2/G_+$ are singled out from those on the whole supergroup $G_1 \otimes G_2$ by requiring them to be invariant under the right gauge G_+ transformations

$$U_j \rightarrow U_j H_j(\xi), \quad H_j(\xi) = \exp\left(\frac{i}{2} y^\mu(\xi) R_\mu^j\right), \quad j = 1, 2. \quad (3.7)$$

One may fix the gauge so that $G_1 \otimes G_2/G_+$ is parametrized by the $N = 2$ superspace type coordinates

$$G_1 \otimes G_2/G_+ = \left\{ x^\mu = \frac{1}{2}(x^{1\mu} - x^{2\mu}), \theta^{1\alpha}, \theta^{2\beta} \right\} \quad (3.8)$$

associated with the coset generators $R_\mu^- = \frac{1}{2}(R_\mu^1 - R_\mu^2)$, S_α^1, S_β^2 . Note that the superspace $G_1 \otimes G_2/G_+$ (like its flat prototype) is not symmetric: (anti)commutators of the coset generators contain in their r.h.s. the generators of the same sort, alongside with those of the stability subgroup G_+ . The supergroups G_1 and G_2 are realized on the elements of the coset $G_1 \otimes G_2/G_+$ by left shifts which are accompanied, in the particular gauge (3.8), by induced right G_+ transformations. As distinct from the flat abelian case, the subgroup G_+ has a nontrivial action on coordinates (3.8). Therefore, the product $G_1 \otimes G_2$ is by no means reduced to anything like $N = 2$ supergroup while realized on these coordinates.

The next step is the construction of Cartan's forms. Like in Sec.2 we first define the left-invariant one-forms on supergroups G_1 and G_2 separately

$$U_j^{-1} \partial_a U_j d\xi^a = \omega^j = \omega_a^j d\xi^a = (\omega_a^{j\mu} R_\mu^j + \omega_a^{j\alpha} S_\alpha^j) d\xi^a \quad (3.9)$$

(no summation over $j = 1, 2$!). They are invariant under global left G_1 and G_2 transformations

$$\begin{aligned} U_j &\rightarrow \exp(i \varepsilon^{j\alpha} S_\alpha^j) U_j \\ U_j &\rightarrow \exp(i a^{j\mu} R_\mu^j) U_j \end{aligned} \quad (3.10)$$

which are a nonabelian generalization of $N = 1$ supertranslations and ordinary translations. Under the gauge transformations (3.7) ω^j transform as gauge connections

$$\omega^j \rightarrow H_j^{-1} \omega^j H_j + H_j^{-1} dH_j \quad (3.11)$$

In what follows, it will be convenient to relate these one-forms to the same set of generators using the fact that (R_μ^1, S_α^1) and (R_μ^2, iS_α^2) obey the same structure relations. One has

$$\omega^1 = \omega^{1\mu} R_\mu + \omega^{1\alpha} S_\alpha, \quad \omega^2 = \omega^{2\mu} R_\mu - i\omega^{2\alpha} S_\alpha \quad (3.12)$$

$$\omega^{i\mu} R_\mu \rightarrow H^{-1} \omega^{i\mu} R_\mu H + H^{-1} dH$$

$$\omega^{i\alpha} S_\alpha \rightarrow H^{-1} \omega^{i\alpha} S_\alpha H, \quad H = \exp\left(i \frac{y^\mu(\xi)}{2} R_\mu\right) \quad (3.13)$$

Note that the structure of U_1 and U_2 implies that ω^1, ω^2 are related via simple changes of supergroup coordinates

$$\begin{aligned} \omega^{1\mu} &\leftrightarrow \omega^{2\mu}, & x^{1\mu} &\leftrightarrow x^{2\mu} \\ \omega^{1\alpha} &\leftrightarrow -i\omega^{2\alpha}, & \theta^{1\alpha} &\leftrightarrow -i\theta^{2\alpha} \end{aligned} \quad (3.14)$$

For future use, we write down the Maurer–Cartan equations which follow from the definitions (3.9) of ω^j

$$d\omega^j + \omega^j \wedge \omega^j = 0 \quad (3.15)$$

or, in terms of the coefficients of one-forms,

$$\begin{aligned} \partial_a \omega_b^{1\mu} - \partial_b \omega_a^{1\mu} + \omega_a^{1\rho} \omega_b^{1\nu} t_{\rho\nu}^\mu + \omega_a^{1\alpha} \omega_b^{1\beta} \Gamma_{\alpha\beta}^\mu &= 0, \\ \partial_a \omega_b^{1\alpha} - \partial_b \omega_a^{1\alpha} + (\omega_a^{1\mu} \omega_b^{1\beta} - \omega_b^{1\mu} \omega_a^{1\beta}) C_{\mu\beta}^\alpha &= 0 \end{aligned} \quad (3.15a)$$

$$\begin{aligned} \partial_a \omega_b^{2\mu} - \partial_b \omega_a^{2\mu} + \omega_a^{2\rho} \omega_b^{2\nu} t_{\rho\nu}^\mu - \omega_a^{2\alpha} \omega_b^{2\beta} \Gamma_{\alpha\beta}^\mu &= 0, \\ \partial_a \omega_b^{2\alpha} - \partial_b \omega_a^{2\alpha} + (\omega_a^{2\mu} \omega_b^{2\beta} - \omega_b^{2\mu} \omega_a^{2\beta}) C_{\mu\beta}^\alpha &= 0. \end{aligned} \quad (3.15b)$$

The one-forms ω^j are the convenient building blocks out of which one may construct the Cartan forms on the coset $G_1 \otimes G_2 / G_+$ and the current one-forms transforming according to the adjoint representation of G_1 and G_2 . The Cartan forms having the homogeneous transformation laws with respect to the right gauge G_+ group

$$\begin{aligned} \omega^{-\mu} R_\mu &= (\omega^{1\mu} - \omega^{2\mu}) R_\mu, \quad \omega^{j\alpha} S_\alpha \\ \omega^{-\mu} R_\mu &\rightarrow H \omega^{-\mu} H^{-1}, \quad \omega^{j\alpha} S_\alpha \rightarrow H \omega^{j\alpha} S_\alpha H^{-1} \end{aligned} \quad (3.16)$$

are interpreted as the covariant differentials of the coset coordinates $(x^\mu, \theta^{1\alpha}, \theta^{2\alpha})$ while the inhomogeneously transforming quantity

$$\begin{aligned} \omega^{+\mu} R_\mu &= \frac{1}{2}(\omega^{1\mu} + \omega^{2\mu}) R_\mu \\ \omega^{+\mu} R_\mu &\rightarrow H^{-1} \omega^{+\mu} R_\mu H + H^{-1} dH \end{aligned} \quad (3.17)$$

defines the G_+ connection.

Further, let us also relate U_1 and U_2 (2.4) to the same set of generators R_μ, S_α and define a G^c matrix field

$$U(\xi) = U_1(\xi)(U_2(\xi))^{-1} = \exp\left(\frac{i}{2}x^{1\mu}R_\mu + i\theta^{1\alpha}S_\alpha\right)\exp\left(-\frac{i}{2}x^{2\mu}R_\mu - \theta^{2\alpha}S_\alpha\right) \quad (3.18)$$

This field is manifestly invariant under G_+ gauge transformations (3.7) (related to the same generators) and so is defined on the coset space $G_1 \otimes G_2/G_+$ (it actually depends on $(x^\mu, \theta^{1\alpha}, \theta^{2\alpha})$). The rigid supergroups G_1 and G_2 act on U as the left and right shifts

$$U' = g_1 U (g_2)^{-1}, \quad g_1 \in G_1, \quad g_2 \in G_2 \quad (3.19)$$

Thus, U is an analog of the chiral field of the principal chiral field sigma models.

The one-forms

$$J^L = U dU^{-1}, \quad J^R = U^{-1} dU = -U^{-1} J^L U \quad (3.20)$$

transform linearly, according to the adjoint representations of G_1 and G_2

$$J^L \rightarrow g_1 J^L (g_1)^{-1}, \quad J^R \rightarrow g_2 J^R (g_2)^{-1} \quad (3.21)$$

These objects (more precisely, their coefficients J_a^L, J_a^R) are similar by the transformation properties to the left and right currents of sigma models for principal chiral field which correspond to the symmetric cosets $G \otimes G/G_{diag}$. Owing to the fact that the coset space $G_1 \otimes G_2/G_+$ is nonsymmetric, we have more freedom in defining the homogeneously transforming objects of the type (3.20). Namely, one may consider the objects

$$\begin{aligned} J^{L-} &= U_1(\omega^{-\mu}R_\mu)U_1^{-1}, \quad J^{Li} = U_1(\omega^{i\alpha}S_\alpha)U_1^{-1}, \\ J^{R-} &= U_2(\omega^{-\mu}R_\mu)U_2^{-1}, \quad J^{Ri} = U_2(\omega^{i\alpha}S_\alpha)U_2^{-1} \end{aligned} \quad (3.22)$$

These quantities are manifestly H -invariant and transform according to the adjoint representations of G_1 and G_2 . The previously defined currents J^L, J^R are specific combinations of (3.22)

$$\begin{aligned} J^L &= -U_1(\omega^1 - \omega^2)U_1^{-1} \equiv -U_1\omega^-U_1^{-1} = \\ &= -U_1[\omega^{-\mu}R_\mu + \omega^{1\alpha} + i\omega^{2\alpha})S_\alpha]U_1^{-1} = -J^{L-} - (J^{L1} + iJ^{L2}) \\ J^R &= U_2\omega^-U_2^{-1} = J^{R-} + (J^{R1} + iJ^{R2}) \end{aligned} \quad (3.23)$$

Note that J^L and J^R , in contrast to the currents of principal chiral field sigma models, do not obey simple conservation laws. The conserved currents generating G_1 and G_2 symmetries are composed from the basic quantities (3.22) in a different way (see subsec 3.5). Nevertheless, J^L and J^R are still useful objects while constructing the invariant actions.

3.3 Invariant WZW action

Now we have all the necessary ingredients to arrange the $G_1 \otimes G_2$ invariant WZW type actions. The simplest invariants are those bilinear in the homogeneously transforming forms $\omega_a^{-\mu}, \omega_a^{j\alpha}$ (covariant derivatives of the coset coordinates). The general expression for this part of the Lagrangian can be written as

$$\mathcal{L}_0 = \sqrt{-g}g^{ab}(\xi)[Str(\omega_a^- \omega_b^-) + \sum_{i,j=1}^2 \kappa_{ij} Str(\omega_a^{i\alpha} S_\alpha \omega_b^{j\beta} S_\beta)] \quad (3.24)$$

with κ_{ij} being constants arbitrary for the moment and $\omega_a^- = \omega_a^1 - \omega_a^2$. In general, this expression is not real and gives rise to the second order derivative kinetic terms both for even $x^\mu(\xi)$ and odd $\theta^{1\alpha}(\xi), \theta^{2\alpha}(\xi)$ coset space coordinates. The first difficulty is circumvented by choosing

$$\kappa_{12} + \kappa_{21} = -2i \quad (3.25)$$

The second peculiarity seems not too fatal while staying at the pure sigma model level ($\theta^{1\alpha}(\xi), \theta^{2\alpha}(\xi)$ are world-sheet scalars). However, keeping in mind going to the abelian GS superstring limit (see Subsec.3.4), such terms are unwanted as they could produce the second order equations of motion for the physical world-sheet fermions in the light-cone gauge. So, we leave in (3.24) only the term bilinear in $\omega_a^{-\mu}$

$$\kappa_{11} = -1, \quad \kappa_{22} = 1, \quad (3.26)$$

$$\mathcal{L}_0 = -\sqrt{-g}g^{ab}\omega_a^{-\mu}\omega_{b\mu}^-. \quad (3.27)$$

It is worth noting that such an opportunity exists only because we deal with the non-symmetric space $G_1 \otimes G_2/G_+$. Owing to this choice the coset Cartan 1-forms $\omega^{-\mu}, \omega^{j\alpha}$ are assigned to a reducible representation of the stability subgroup G_+ and, as a result, are separately covariant. This is a crucial difference from the principal chiral field sigma models where the underlying coset space $G \otimes G/G_{diag}$ is symmetric. There, the Cartan 1-forms valued in the coset constitute an irreducible representation of the stability subgroup G_{diag} . So, only the invariants having the form of the first term in (3.25) are admissible in such models. Note that the WZW model discussed recently by Green [12] belongs to this latter class.

Let us now turn to construction of the relevant WZ term. Once again, due to the fact that $\omega^{-\mu}$ and $\omega^{j\alpha}$ are covariant separately, one has more possibilities for constructing this term compared to the $G \otimes G/G_{diag}$ case. We choose it in the following way

$$WZ = \int_V \Omega_3 = \int_V d^3\xi \varepsilon^{ABC} Str[\omega_A^- \omega_B^- \omega_C^- + \sum_{i,j=1}^2 \tilde{\kappa}_{ij} \partial_A(\omega_B^{i\alpha} S_\alpha \omega_C^{j\beta} S_\beta)] \quad (3.28)$$

with constants κ_{ij} being arbitrary for the moment. It is easy to check that the three-form Ω_3 is invariant under global G_1, G_2 and gauge G_+ transformations and satisfy the standard closeness condition

$$\delta\Omega_3 = d\Omega_2 \quad (3.29)$$

where Ω_2 is a two-form. Actually, these requirements are met by the two pieces in (3.28) separately. The first piece is an analog of the standard $G \otimes G/G_{diag}$ WZ term while the second one is new. The possibility to add it is related to the property that the one-forms $\omega_b^{i\alpha} d\xi^b$ are covariant in their own right.

Leaving aside the question of how general (3.28) is and anticipating the analysis of Subsec.3.4, we point out that the ansatz (3.28) is sufficient for ensuring a correct flat superspace limit of the action. In fact, the constants $\tilde{\kappa}_{ij}$ in (3.28) are severely restricted by the reality condition. Using the MC equations (3.15), it is straightforward to find

$$\begin{aligned} \varepsilon^{ABC} Str(\omega_A^- \omega_B^- \omega_C^-) &= -\varepsilon^{ABC} \left[\frac{1}{2} \omega_A^{-\mu} \omega_B^{-\nu} \omega_C^{-\rho} t_{\mu\nu\rho} + \right. \\ &\quad \left. + \frac{3}{2} (\omega_A^{1\alpha} \omega_B^{1\beta} - \omega_A^{2\alpha} \omega_B^{2\beta} + 2i \omega_A^{1\alpha} \omega_B^{2\beta}) \omega_C^{-\rho} \Gamma_{\rho, \alpha\beta} \right], \end{aligned} \quad (3.30)$$

$$\begin{aligned} \sum_{i,j=1}^2 \tilde{\kappa}_{ij} \varepsilon^{ABC} \partial_A Str(\omega_B^{i\alpha} S_\alpha \omega_C^{j\beta} S_\beta) &= \\ &= (\tilde{\kappa}_{12} - \tilde{\kappa}_{21}) \varepsilon^{ABC} (\omega_C^{-\mu} \omega_A^{1\alpha} \omega_B^{2\beta}) \Gamma_{\mu, \alpha\beta}. \end{aligned} \quad (3.31)$$

Thus, the imaginary part is absent in (3.28) with the choice

$$\tilde{\kappa}_{12} - \tilde{\kappa}_{21} = 3i \quad (3.32)$$

(as is seen from (3.31), parameters $\tilde{\kappa}_{ii}$ and $\tilde{\kappa}_{zz}$ drop out altogether).

The invariant action obtained as a sum of (3.27) and (3.28) involves at this step two free parameters ℓ_I and ℓ_{II} (cf. Eq.(2.16))

$$\begin{aligned} A &= -\ell_I \int_{\partial V} d^2 \xi \sqrt{-g} g^{ab} (\omega_a^{-\mu} \omega_{b\mu}^-) - \\ &\quad - \frac{1}{2} \ell_{II} \int_V d^3 \xi \varepsilon^{ABC} \{ \omega_A^{-\mu} \omega_B^{-\nu} \omega_C^{-\rho} t_{\mu\nu\rho} + 3(\omega_A^{1\alpha} \omega_B^{2\beta} - \omega_A^{2\alpha} \omega_B^{2\beta}) \omega_C^{-\rho} \Gamma_{\rho, \alpha\beta} \} \end{aligned} \quad (3.33)$$

or

$$\begin{aligned} A &= \int_{\partial V} d^2 \xi \left\{ -\ell_I \sqrt{-g} g^{ab} (\omega_a^{-\mu} \omega_{b\mu}^-) - 3\ell_{II} \varepsilon^{ab} (\omega_a^{1\mu} \omega_{b\mu}^2) \right\} + \\ &\quad + \ell_{II} \int_V d^3 \xi \varepsilon^{ABC} \{ \partial_A \omega_B^{1\mu} \omega_{C\mu}^1 - \partial_A \omega_B^{2\mu} \omega_{C\mu}^2 + \\ &\quad + (\omega_A^{1\alpha} \partial_B \omega_C^{1\beta} + \omega_A^{2\alpha} \partial_B \omega_C^{2\beta}) X_{\alpha\beta} \} \end{aligned} \quad (3.33')$$

(the second representation is more convenient when varying A). In order to fix ℓ_I and ℓ_{II} , one needs to consider the flat superspace limit of (3.33).

3.4 The $N = 2$ GS type superstrings as the limiting case of the $G_1 \otimes G_2/G_+$ WZW sigma models

We accept a natural requirement that (3.33) goes over to the $N=2$ GS superstring type action (2.18) in the flat limit, when the superalgebras (3.1), (3.3) contract into a sum of $N=1$ supertranslation algebras

$$\begin{aligned}\{Q_\alpha^1, Q_\beta^1\} &= -\Gamma_{\alpha\beta}^\mu P_\mu^1, [Q_\alpha^1, P_\mu^1] = [P_\mu^1, P_\nu^1] = 0, \\ \{Q_\alpha^2, Q_\beta^2\} &= \Gamma_{\alpha\beta}^\mu P_\mu^2, [Q_\alpha^2, P_\mu^2] = [P_\mu^2, P_\nu^2] = 0\end{aligned}\quad (3.34)$$

We call this case “abelian” in view of commutativity of bosonic generators. As was explained in Sec.2, in general $\Gamma_{\alpha\beta}^\mu$ in (3.34) do not necessarily coincide with Dirac matrices figuring in the genuine $N = 2$ GS action [10]. It is also worth recalling that the contracted dual superalgebras are isomorphic to each other, as opposed to their nonabelian prototypes (3.1), (3.3).

The contraction of (3.1) and (3.3) can be effected by introducing a parameter c

$$R_\mu = c^2 P_\mu, \quad S_\alpha = c Q_\alpha, \quad (3.35)$$

$$\{Q_\alpha, Q_\beta\} = -\Gamma_{\alpha\beta}^\mu P_\mu, [P_\mu, Q_\alpha] = \frac{1}{c^2} C_{\mu\alpha}^\beta Q_\beta, [P_\mu, P_\nu] = \frac{1}{c^2} t_{\mu\nu}^\lambda P_\lambda \quad (3.36)$$

and then taking the limit $c \rightarrow \infty$ in (3.36). For Cartan’s 1-forms and the action the contraction procedure involves rescaling the target superspace coordinates

$$\tilde{\theta}^{i\alpha} = c \theta^{i\alpha}, \quad \tilde{x}^{j\mu} = c^2 x^{j\mu} \quad (3.36a)$$

and passing to the generators P_μ, Q_α . One gets

$$\begin{aligned}\omega_a^{j\mu} &= \frac{1}{c^2} \frac{i}{2} (\partial_a \tilde{x}^{j\mu} - (-)^j i \partial_a \tilde{\theta}^j \Gamma^\mu \tilde{\theta}^j) + 0(\frac{1}{c^4}), \\ \omega_a^{j\alpha} &= \frac{i}{c} [\partial_a \tilde{\theta}^{j\alpha} - \frac{i}{4c^2} (\partial_a \tilde{\theta}^{j\beta} \tilde{x}^{j\mu} - \partial_a \tilde{x}^{j\mu} \tilde{\theta}^{j\beta} + \\ &\quad + (-)^j \frac{i}{3} (\partial_a \tilde{\theta}^j \Gamma^\mu \tilde{\theta}^j) \tilde{\theta}^{j\beta}) C_{\mu\beta}^\alpha] + 0(1/c^5).\end{aligned}\quad (3.37)$$

Substituting these expressions into (3.33) and keeping in the action the leading terms in $\frac{1}{c}$ ($\sim \frac{1}{c^4}$) it is straightforward to find ($\tilde{x}^\mu = \tilde{x}^{1\mu} - \tilde{x}^{2\mu}$)

$$\begin{aligned}A &= \frac{(-1)}{4c^4} \int_{\partial V} d^2 \xi \{ -\ell_I \sqrt{-g} g^{ab} (\partial_a \tilde{x}^\mu + i \sum_{j=1}^2 \partial_a \tilde{\theta}^j \Gamma^\mu \tilde{\theta}^j) (\partial_b \tilde{x}_\mu + i \sum_{j=1}^2 \partial_b \tilde{\theta}^j \Gamma_\mu \tilde{\theta}^j) - \\ &\quad - 3\ell_{II} \varepsilon^{ab} [i \partial_a \tilde{x}^\mu - \frac{1}{2} \sum_{j=1}^2 \partial_a \tilde{\theta}^j \Gamma^\mu \tilde{\theta}^j] (\sum_{k,j=1}^2 \partial_b \tilde{\theta}^k \sigma_3^{kj} \Gamma_\mu \tilde{\theta}^j) \} + 0(\frac{1}{c^6})\end{aligned}\quad (3.38)$$

Comparing (3.38) with the GS superstring type action [10] (2.18) one concludes that they coincide (up to an inessential numerical coefficient $(-1)/4c^4$) iff

$$\ell_{II} = -\frac{2}{3}\ell_I \quad (3.39)$$

(in case of $\Gamma_{\alpha\beta}^\mu$ being Dirac γ -matrices, one gets the genuine $N = 2$ GS covariant action (2.18)).

At this step it is appropriate to explain why for constructing a nonabelian superstring action one has to start with the self-dual supergroup $G = G_1 \otimes G_2$ rather than the product of two isomorphic supergroups. The second option corresponds to the substitution $\theta^{2\alpha} \rightarrow i\theta^{2\alpha}$ in (3.38), so in the flat superspace limit one would have, e.g. $\partial_a \theta^1 \Gamma^\mu \theta^1 - \partial_a \theta^2 \Gamma^\mu \theta^2$ instead of the correct GS expression $\partial_a \theta^1 \Gamma^\mu \theta^1 + \partial_a \theta^2 \Gamma^\mu \theta^2$. Such a noncompact version of $N=2$ GS superstring (it is based on $N = 2$ supersymmetry with the $SO(1, 1)$ automorphism group) would contain ghost fermionic degrees of freedom in the light-cone gauge.

The final expression for the invariant action of our $G_1 \otimes G_2/G_+$ WZW sigma model is as follows

$$\begin{aligned} A = \ell_I \{ & \int_{\partial V} d^2 \xi [-\sqrt{-g} g^{ab} \omega_a^{-\mu} \omega_{b\mu}^- + 2\varepsilon^{ab} (\omega_a^{1\mu} \omega_{b\mu}^2)] - \\ & - \frac{2}{3} \int_V d^3 \xi \varepsilon^{ABC} [\partial_A \omega_B^{1\mu} \omega_{C\mu}^1 - \partial_A \omega_B^{2\mu} \omega_{C\mu}^2 + \\ & + (\omega_A^{1\alpha} \partial_B \omega_C^{1\beta} + \omega_A^{2\alpha} \partial_B \omega_C^{2\beta}) X_{\alpha\beta}] \} \end{aligned} \quad (3.40)$$

Remarkably, when all the fermionic fields are put equal to zero, (3.40) is reduced to the familiar conformally invariant WZW action for the even subgroup G (this is seen most directly when substituting the relation (3.39) into the action written in the form (3.33)). Thus, (3.40) can be regarded as a genuine $N = 2$ superstring extension of the group manifold string action.

3.5 The equations of motion. Zero-curvature representation

Being written through the covariant variations

$$\tilde{\omega}^j = U_j^{-1} \delta U_j = \tilde{\omega}^{j\mu} R_\mu^j + \tilde{\omega}^{j\alpha} S_\alpha^j \quad (3.41)$$

$$\delta \omega_a^j - \partial_a \tilde{\omega}^j = [\omega_a^j, \tilde{\omega}^j] \quad (3.42)$$

the full variation of the action (3.40) is given by

$$\begin{aligned} \delta A = \ell_I \int_{\partial V} d^2 \xi [& -\delta(\sqrt{-g} g^{ab}) \omega_a^{-\mu} \omega_{b\mu}^- + 2(\partial_a (P_-^{ab} \omega_b^{1\mu} - P_+^{ab} \omega_b^{2\mu}) - \\ & - P_+^{ab} \omega_a^{1\lambda} \omega_b^{2\rho} t_{\lambda\rho}^\mu) \tilde{\omega}_\mu^- - 2P_+^{ab} \omega_{b\mu}^- (\omega_a^{1\alpha} \Gamma_{\alpha\beta}^\mu \tilde{\omega}^{1\beta}) - \\ & - 2P_-^{ab} \omega_{b\mu}^- (\omega_a^{2\alpha} \Gamma_{\alpha\beta}^\mu \tilde{\omega}^{2\beta})] , \end{aligned} \quad (3.43)$$

where $\tilde{\omega}_\mu = \tilde{\omega}_\mu^1 - \tilde{\omega}_\mu^2$ and $P_\pm^{ab} = \sqrt{-g}g^{ab} \pm \varepsilon^{ab}$ [6]. In deducing (3.43), we have essentially used the equations (3.15), (3.42) and Jacobi identities for the structure constants.

The equations of motion following from (3.43) are written as

$$\partial_a(P_-^{ab}\omega_b^{1\mu} - P_+^{ab}\omega_b^{2\mu}) - t_{\lambda\rho}^\mu P_+^{ab}\omega_a^{1\lambda}\omega_b^{2\rho} = 0 \quad (3.44)$$

$$P_+^{ab}\omega_a^{1\beta}\Gamma_{\alpha\beta}^\mu\omega_{b\mu}^- = 0, P_-^{ab}\omega_a^{2\beta}\Gamma_{\beta\alpha}^\mu\omega_{b\mu}^- = 0, \quad (3.45)$$

$$\omega_a^{-\mu}\omega_{b\mu}^- - \frac{1}{2}g_{ab}g^{cd}\omega_c^{-\mu}\omega_{d\mu}^- = 0 \quad (3.46)$$

The last equation is obtained by varying the world-sheet metric g^{ab} and is nothing else than the classical version of standard Virasoro constraints. Equation (3.44) is the equation of motion for $x^\mu(\xi)$ while Eqs. (3.45) are those for fermions $\theta^{1\alpha}(\xi), \theta^{2\alpha}(\xi)$. For completeness, one should also add to (3.44)–(3.45) the MC equations (3.15). Then this extended system gives a closed set of equations for the $G_1 \otimes G_2/G_+$ Cartan 1-forms. So, in our approach the latter can be treated as the primary independent objects (cf. [14, 15], where analogous equations entirely in terms of Cartan's forms have been obtained for ordinary bosonic sigma models, including those with the WZ terms [15]).

With exploiting the MC equations, the set (3.44), (3.45) can be given different equivalent representations. One may, e.g. rewrite it as the two systems

$$(\nabla_a^1)^\mu_\lambda (P_+^{ab}\omega_b^{-\lambda}) + \varepsilon^{ab}\omega_a^{1\alpha}\Gamma_{\alpha\beta}^\mu\omega_b^{1\beta} = 0, \quad (3.47a)$$

$$P_+^{ab}\omega_{b\mu}^-\Gamma_{\alpha\beta}^\mu\omega_a^{1\beta} = 0$$

$$(\nabla_a^2)^\mu_\lambda (P_-^{ab}\omega_b^{-\lambda}) - \varepsilon^{ab}\omega_a^{2\alpha}\Gamma_{\alpha\beta}^\mu\omega_b^{2\beta} = 0, \quad (3.47b)$$

$$P_-^{ab}\omega_{b\mu}^-\Gamma_{\alpha\beta}^\mu\omega_a^{2\beta} = 0,$$

where $(\nabla_a^j)^\mu_\lambda = \delta_\lambda^\mu\partial_a + t_{\nu\lambda}^\mu\omega_a^{j\nu}$ are covariant derivatives with respect to gauge transformations (3.13). Remarkably, like in the GS superstring case (Eqs.(2.23)) one may cast (3.47) into a simple form as the conservation laws for the appropriate linearly transforming currents (see Eq.(3.22)).

$$\partial_a(P_+^{ab}J_b^{L-} + \varepsilon^{ab}J_b^{L1}) = 0, \quad (3.47a')$$

$$\partial_a(P_-^{ab}J_b^{R-} - i\varepsilon^{ab}J_b^{R2}) = 0 \quad (3.47b')$$

Just these combinations of currents generate global G_1 and G_2 symmetries of the action (3.40).

The property that the equations of motion are divided into the two sets, respectively for the “left” and “right” variables $\{P_+^{ab}\omega_b^{-\lambda}, \omega_a^{1\alpha}\}, \{P_-^{ab}\omega_b^{-\lambda}, \omega_a^{2\alpha}\}$ seems to reflect the product structure of the underlying supergroup $G = G_1 \otimes G_2$. So we expect G_1 to be eventually realized on the left variables while G_2 on the right ones. Of course, this statement can be given a precise meaning only on the solutions of these equations, e.g. after passing to a light-cone type gauge, and upon performing quantization.

A remarkable property of the obtained equations is their complete integrability.

A zero curvature representation for Eqs.(3.47) is written as

$$[L_a^+, L_b^+] = [L_a^-, L_b^-] = 0 \quad (3.48)$$

where

$$\begin{aligned} L_a^+ &= \partial_a - \frac{1}{2}(1 - \lambda^2)\varepsilon_{ab}P_+^{bc}J_c^{L-} - (\lambda + 1)J_a^{L1} \\ L_a^- &= \partial_a - \frac{1}{2}(1 - \lambda^2)\varepsilon_{ab}P_-^{bc}J_c^{R-} + i(\lambda + 1)J_a^{R2} \end{aligned} \quad (3.49)$$

and the currents J_c^{L-} , J_c^{R-} , J_a^{L1} , J_a^{R2} have been defined in Eqs.(3.22). Here, the generators R_μ, S_α satisfy the superalgebra (3.1); λ is a spectral parameter.

The vanishing of the first and second commutators in (3.48) yields, respectively, Eqs.(3.47a) and (3.47b) (to be more precise, their current form (3.47a') and (3.47b')). The MC equations (3.15) are also encoded in the integrability conditions (3.48). Note that L_a^+ is in fact defined on the super-algebra (3.1) while L_a^- on (3.3) (one could define L_a^+, L_a^- on the two sets of mutually commuting abstract generators R_μ^1, S_α^1 and R_μ^2, S_α^2 obeying (3.1) and (3.3); in (3.49) for simplicity we have related L_a^+, L_a^- to the same set of generators).

The possibility to represent the equations of motion in the form (3.48), (3.49) leads us to the conclusion that we have constructed a new completely integrable 2D system. Thus, the integrability of the abelian $N = 2$ GS superstring type models (including genuine $N = 2$ GS superstring) mentioned in Sec.1 turns out a particular case of a more general phenomenon inherent in the nonabelian models. As usual, the integrability suggests the existence of infinitely many conserved currents which can be evaluated using an auxiliary spectral problem associated with the operators L_1^\pm (see e.g. [16]). Also, this is an indication that the set of Eqs. (3.47) can be linearized in proper variables. Let us emphasize a crucial role of the WZ term for attaining the integrability. We have checked that without this term the equations of motion can never be brought into the form (3.48). Moreover, the integrability comes out only if the free parameters ℓ_I and ℓ_{II} are adjusted so as to give the action (3.40). This is in accordance with the general statement of [17] about nonintegrability of ordinary (i.e. having no WZ terms) sigma models defined on nonsymmetric coset spaces. Our consideration demonstrates that in a number of cases the integrability can be achieved by adding proper WZ terms to the sigma model action. It seems that this phenomenon is not specific for supergroups only. It would be interesting to study its implications in purely bosonic 2D sigma models.

For completeness, we present the equations of motion (3.44), (3.45) and the MC equations (3.15) in the conformal gauge

$$\begin{aligned} \sqrt{-g}g^{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1), \quad \omega_\pm = \frac{1}{2}(\omega_0 \pm \omega_1), \\ \partial_+\omega_-^{1\mu} - \partial_-\omega_+^{2\mu} - t_{\lambda\rho}^\mu \omega_-^{1\lambda} \omega_+^{2\rho} &= 0, \\ \Gamma_{\alpha\beta}^\mu \omega_{+\mu}^- \omega_-^{1\alpha} &= 0, \end{aligned}$$

$$\Gamma_{\alpha\beta}^{\mu}\omega_{-\mu}^{-}\omega_{+}^{2\alpha}=0 \quad (3.50)$$

$$\begin{aligned} \partial_{+}\omega_{-}^{1\mu}-\partial_{-}\omega_{+}^{1\mu}+t_{\nu\lambda}^{\mu}\omega_{+}^{1\nu}\omega_{-}^{1\lambda}+\Gamma_{\alpha\beta}^{\mu}\omega_{+}^{1\alpha}\omega_{-}^{1\beta}&=0, \\ \partial_{+}\omega_{-}^{2\mu}-\partial_{-}\omega_{+}^{2\mu}+t_{\nu\lambda}^{\mu}\omega_{+}^{2\nu}\omega_{-}^{2\lambda}-\Gamma_{\alpha\beta}^{\mu}\omega_{+}^{2\alpha}\omega_{-}^{2\beta}&=0, \\ \partial_{+}\omega_{-}^{j\alpha}-\partial_{-}\omega_{+}^{j\alpha}+C_{\mu\beta}^{\alpha}(\omega_{+}^{j\mu}\omega_{-}^{j\beta}-\omega_{+}^{j\beta}\omega_{-}^{j\mu})&=0 \end{aligned} \quad (3.51)$$

3.6 κ -supersymmetry

Our last topic will be exploring the conditions under which the action (3.40) possesses a local fermionic κ -symmetry [6, 18].

The only additional requirement needed for ensuring such a symmetry is the existence of matrices $(\tilde{\Gamma}^{\mu})^{\beta\gamma}$ satisfying the relations

$$\Gamma_{\alpha\beta}^{\mu}(\tilde{\Gamma}^{\nu})^{\beta\gamma}+\Gamma_{\alpha\beta}^{\nu}(\tilde{\Gamma}^{\mu})^{\beta\gamma}=2\eta^{\mu\nu}\rho_{\alpha}^{\gamma}, \quad (3.52)$$

where ρ_{α}^{γ} is an arbitrary matrix which can be degenerate.

Using the general expression for δA (3.43) it is easy to show that δA vanishes under the following variations

$$\begin{aligned} \tilde{\omega}^{-\mu}&=0 \\ \tilde{\omega}^{1\alpha}&=P_{+}^{ab}\omega_{b\mu}^{-}(\tilde{\Gamma}^{\mu})^{\alpha\beta}\kappa_{a\beta}^1(\xi), \\ \tilde{\omega}^{2\alpha}&=P_{-}^{ab}\omega_{b\mu}^{-}(\tilde{\Gamma}^{\mu})^{\alpha\beta}\kappa_{a\beta}^2(\xi) \\ \delta(\sqrt{-g}g^{ab})&=-2(P_{+}^{da}P_{+}^{cd}(\omega_d^{1\beta}\rho_{\beta}^{\alpha}\kappa_{c\alpha}^1)-P_{-}^{da}P_{-}^{cb}(\omega_d^{2\beta}\rho_{\beta}^{\alpha}\kappa_{c\alpha}^2)) \end{aligned} \quad (3.53)$$

Here $\kappa_{a\beta}^{1,2}(\xi)$ are the odd transformation parameters. These transformations go over to the standard κ -symmetry transformations of the ordinary $N=2$ GS superstring in the flat superspace limit (when $\Gamma_{\alpha\beta}^{\mu}C^{\beta\gamma}$ coincide with Dirac γ matrices) and thus generalize this symmetry to nonabelian case. Like in the case of the GS superstring model, due to the presence of local κ -supersymmetry, the theory in question is expected to have a representation via free fields (in a suitable gauge). Thus, if Eq.(3.52) has a solution, the nonabelian superstring model seems to be reducible to a 2D conformal field theory. Otherwise, local κ -supersymmetry is lacking and an equivalence of the full action (3.40) to a conformally-invariant one becomes questionable (though conformal invariance still persists for the bosonic part of (3.40)). Fortunately, even in this case our model is expected to admit a kind of linearization because of its complete integrability. Whether this possibility is still related to some infinite-dimensional (local) supersymmetry hidden in the zero curvature representation (3.48) is an interesting open question.

Finally, let us emphasize that the condition (3.52) places severe limitations on the appropriate class of superalgebras (3.1). It is easy to see that, in the case of a nondegenerate matrix ρ , Eq.(3.52) is the defining relation of some Clifford algebra. Indeed, (3.52)

implies that $\tilde{\Gamma}^\nu \rho^{-1}$ like Γ^ν is a symmetric matrix. So, Γ^ν and $\tilde{\Gamma}^\nu$ can be combined into a single matrix ⁵

$$\gamma^\mu = \begin{pmatrix} 0 & \Gamma^\mu \\ \tilde{\Gamma}^\mu \rho^{-1} & 0 \end{pmatrix} \quad (3.54)$$

which satisfies, as a consequence of (3.52), the standard Clifford algebra relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \quad (3.55)$$

(Γ^μ and $\tilde{\Gamma}^\mu \rho^{-1}$ are a kind of Weyl projections of γ^μ). If $\Gamma_{\alpha\beta}^\mu$ provides an irreducible representation of (3.55), we have severe restrictions on the dimensions of generators entering into the superalgebra (3.1). Otherwise, $\Gamma_{\alpha\beta}^\mu$ have a block structure, each block corresponding to an irreducible representation of Clifford algebra (3.55). In view of the general cyclic identity (3.2), for each block one also has the well-known restrictive relations between the dimensions of vector and spinor indices. In the case of a degenerate matrix ρ we are led to reduce the space on which this matrix acts, after which we are again left with the nondegenerate situation discussed above. An example of the superalgebra with a nondegenerate metric for which one may find $\tilde{\Gamma}^\nu$ (3.52) corresponding to a degenerate matrix ρ is given by the Green superalgebra [12]. It can be cast into the form (3.1) by joining its supertranslation generator Q_α and the extra fermionic central charge generator K^α into a single generator $S_{\hat{\alpha}}$. For this case we have

$$\Gamma^\mu = \tilde{\gamma}^\mu \sim \begin{pmatrix} \Gamma_1^\mu & 0 \\ 0 & 0 \end{pmatrix}$$

and Γ_1^μ are matrices which are related to Dirac γ matrices and satisfy the identity (3.2). It would be interesting to inquire whether there exist nontrivial superalgebras (3.1) of that sort having a nonabelian even part.

4 EXAMPLES ⁶

Here we illustrate the general consideration of previous Sections by simple examples of nonabelian $N = 2$ superstrings. All these examples reveal local fermionic κ -invariance, thus demonstrating that the variety of the κ -invariant nonabelian $N = 2$ superstring models is not empty.

Actually, listing all possible models of that sort amounts to classifying all possible superalgebras (3.1) which have a non-degenerate Killing supermetric and whose structure constants $\Gamma_{\alpha\beta}^\mu$ obey the Clifford algebra condition (3.52). In other words, one needs to classify the superalgebras with the structure constants $\Gamma_{\alpha\beta}^\mu$ forming a representation (irreducible or reducible) of Clifford algebra. We have no a general solution to this interesting

⁵We thank P.S. Howe for suggesting this to us.

⁶This Section (together with refs. [21] - [23]) is absent in the original ICTP preprint. It has been prepared for a revised version of the paper submitted to a journal. We expose it here also without updating.

algebraic problem as yet. However, now we are aware of a number of particular examples of such superalgebras which can be used to construct new non-trivial superstring type models.

The list of κ -invariant models known to us involves those associated with the superalgebras $osp(2|1)$, $su(2|1)$, as well as with various direct sums of these superalgebras which may in addition include the Green superalgebra [12]. As was already mentioned, the latter also belongs to the type we are interested in.

In order to be able to relate the considered models to special solutions of a curved background, it is advantageous to rewrite the general action (3.40) as the action for the superstring moving in a $N = 2$ supergravity background [19]

$$\begin{aligned} A = & l_I \left\{ \int_{\partial V} d^2 \xi \left[-\frac{1}{2} \Phi(z(\xi)) \eta_{\mu\nu} \sqrt{-g} g^{ab} (\partial_a z^M E_M^\mu) (\partial_b z^N E_N^\nu) \right] + \right. \\ & \left. + \frac{1}{2} \int_V d^3 \xi \varepsilon^{abc} \partial_a z^M \partial_b z^N \partial_c z^K H_{MNK} \right\}. \end{aligned} \quad (4.1)$$

Here $\Phi(z)$ is the dilaton superfield which appears in the superfield formulations of the type II supergravities (one can get rid of this superfield by rescaling $E_M^\mu \rightarrow \frac{1}{\sqrt{\Phi}} E_M^\mu$), $z^M = \{x^\mu, \theta^{1\alpha}, \theta^{2\alpha}\}$ are coordinates of the coset space $G_1 \times G_2 / G_+$ (we fix the right gauge freedom by the condition $x^{1\mu} + x^{2\mu} = 0$ and thus put $x^\mu = \frac{1}{2}(x^{1\mu} - x^{2\mu}) = x^{1\mu} = -x^{2\mu}$) and $E_M^\mu = \frac{1}{\sqrt{\Phi}} (E_M^{1\mu} - E_M^{2\mu})$ where $E_M^{j\mu}$ are related to the previously used quantities as

$$\omega_a^{j\mu} = \partial_a z^M E_M^{j\mu}.$$

Finally, the three-index field strength H_{MNK} is related to the three-form (see eq.(3.33))

$$\begin{aligned} H &= dz^M dz^N dz^K H_{MNK} = \\ &= \frac{1}{3} \Phi^{3/2} t_{\mu\nu\lambda} E^\mu E^\nu E^\lambda + \Phi^{1/2} \Gamma_{\mu,\alpha\beta} (E^\mu E^{1\alpha} E^{1\beta} - E^\mu E^{2\alpha} E^{2\beta}), \end{aligned} \quad (4.2)$$

where

$$\begin{aligned} t_{\mu\nu\lambda} &= -\eta_{\mu\rho} t_{\nu\lambda}^\rho, \quad \Gamma_{\mu,\alpha\beta} = \eta_{\mu\nu} \Gamma_{\alpha\beta}^\nu, \quad E^A = (\Phi^{-1/2} (E^{1\mu} - E^{2\mu}), E^{1\alpha}, E^{2\alpha}) \\ E^{j\mu} &= dz^M E_M^{j\mu}, \quad E^{j\alpha} = dz^M E_M^{j\alpha} \equiv d\xi^a \omega_a^{j\alpha}. \end{aligned} \quad (4.3)$$

Let us now turn to a brief description of our examples.

4.7 The $OSp(2|1, R)$ model

The superalgebra $osp(2|1, R)$ includes three even and two odd generators which obey the (anti)commutation relations of the form (3.1) with

$$C_\mu = \left\{ -\frac{i}{2} \sigma_2, \frac{1}{2} \sigma_3, \frac{1}{2} \sigma_1 \right\} = \Gamma_\mu X \quad (4.4a)$$

$$\Gamma^\mu = \left\{ \frac{i}{2} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{i}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad X^{\alpha\beta} = i\varepsilon^{\alpha\beta}, \quad \varepsilon^{12} = 1 \quad (4.4b)$$

$$\eta_{\mu\nu} = -Str(R_\mu R_\nu) = diag(1, -1, -1), \quad t_{\mu\nu\rho} = -\eta_{\mu\lambda} t_{\nu\rho}^\lambda = \varepsilon_{\mu\nu\rho}. \quad (4.4c)$$

Here σ_μ are Pauli matrices and $\varepsilon_{123} = 1$. Note that the even subalgebra of $osp(2|1, R)$ is $sl(2, R) \sim so(1, 2)$.

Due to the property that Γ^μ are related to Pauli matrices, it is easy to construct the matrices $\tilde{\Gamma}^\mu$ satisfying eq.(3.52) with $\rho = 1$

$$\tilde{\Gamma}^\mu = \left\{ 2i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 2i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \quad (4.5)$$

Thus, the nonabelian sigma model (4.1) with the target superspace

$$OSp(2|1) \times OSp(2|1)^*/SL(2, R)_{diag}$$

possesses fermionic κ -invariance and hence can be interpreted as the model of $N = 2$ superstring moving in a special $D = 3$, $N = 2$ supergravity background. Comparing the particular value of the field strength H_{MNK} in (4.2) for the choice (4.4c) with the general expression for H_{MNK} in $D = 3$, $N = 1$ Poincare supergravity [21], we find that the nonvanishing tangent space coefficients of this strength are the same in both cases. This coincidence leads us to the conclusion that the above background supplies an $N = 2$ extension of some particular solution to $D = 3$, $N = 1$ Poincare supergravity [22], with $OSp(2|1)$ as the isometry supergroup.

Being κ -invariant, the model under consideration is expected to be conformally-invariant and to possess world-sheet supersymmetry. The latter can be argued by counting essential bosonic and fermionic degrees of freedom. After imposing the light-cone gauge there remain $2(D - 2) = 2$ bosonic degrees, one left and one right movers. Accordingly, two of four originally present fermionic degrees of freedom are gauged away by κ -symmetry, so that the final number of fermionic coordinates equals 2. Thus, the number of unremovable bosonic variables matches with that of fermionic ones, which is a strong indication of hidden 2D world-sheet supersymmetry. In turn this suggests that the superstring model we deal with is equivalent to some fermionic string model.

4.8 The $SU(2|1)$ model

The superalgebra $su(2|1)$ includes four even generators (R_μ , $\mu = 0, 1, 2, 3$) which form the algebra $u(2) = u(1) + su(2)$ and four odd generators (a complex $SU(2)$ doublet $\Gamma_{\hat{\alpha}}$, $\bar{\Gamma}^{\hat{\beta}} \equiv (\Gamma_{\hat{\beta}})^\dagger$; $\hat{\alpha}, \hat{\beta}, \dots = 1, 2$).

The (anti)commutation relations of this superalgebra can be found in [23]. We will be interested here mainly in the anticommutation relations which can be written as

$$\{\Gamma_{\hat{\alpha}}, \bar{\Gamma}^{\hat{\beta}}\} = (\tau^\mu)_{\hat{\alpha}}^{\hat{\beta}} R_\mu, \quad \{\Gamma_{\hat{\alpha}}, \Gamma_{\hat{\beta}}\} = \{\bar{\Gamma}^{\hat{\alpha}}, \bar{\Gamma}^{\hat{\beta}}\} = 0, \quad (4.6)$$

where

$$(\tau^\mu)_{\hat{\alpha}}^{\hat{\beta}} = \left\{ \delta_{\hat{\alpha}}^{\hat{\beta}}, (\sigma_1)_{\hat{\alpha}}^{\hat{\beta}}, (\sigma_2)_{\hat{\alpha}}^{\hat{\beta}}, (\sigma_3)_{\hat{\alpha}}^{\hat{\beta}} \right\}. \quad (4.7)$$

Combining Γ and $\bar{\Gamma}$ into a single self-conjugated spinor $S_\alpha = (\Gamma_{\hat{\alpha}}, \bar{\Gamma}^{\hat{\beta}})$ ($\alpha, \beta, \dots = 1, 2, 3, 4$) we represent eqs.(4.6) in the form

$$\{S_\alpha, S_\beta\} = \Gamma_{\alpha\beta}^\mu R_\mu, \quad (4.8)$$

where (the superscript T means matrix transposition)

$$\Gamma^\mu = \left\{ \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3^T & 0 \end{pmatrix} \right\}. \quad (4.9)$$

Now the whole set of the (anti)commutation relations of $su(2|1)$ can be given the generic form (3.1) with

$$t_{ik}^l = \varepsilon_{ikl}, \quad t_{0\mu}^\lambda = t_{\mu\nu}^0 = 0, \quad X = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

It is straightforward to find the matrices $\tilde{\Gamma}^\mu$ which satisfy eq.(3.52) with $\eta_{\mu\nu} = -Str(R_\mu^{ad} R_\nu^{ad}) = diag(1, -1, -1, -1)$

$$\tilde{\Gamma}^\mu = \left\{ \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sigma_1^T \\ -\sigma_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sigma_2^T \\ -\sigma_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sigma_3^T \\ -\sigma_3 & 0 \end{pmatrix} \right\}. \quad (4.10)$$

Let us remark that the even $su(2|1)$ generators in adjoint representation R_μ^{ad} are given by the following 6×6 supermatrices

$$R_0^{ad} = \frac{1}{2} \left(\begin{array}{c|cc} 0 & & 0 \\ \hline & -I & 0 \\ \hline 0 & & I \end{array} \right), \quad R_j^{ad} = \frac{1}{2} \left(\begin{array}{c|cc} 0 & & 0 \\ \hline 0 & -2i\varepsilon_{jkl} & 0 \\ \hline 0 & & \sigma_j & 0 \\ & & 0 & -\sigma_j \end{array} \right), \quad j, k, l = 1, 2, 3$$

and we meet here an interesting situation when the even subgroup of $SU(2|1)$ is compact, $SU(2) \times U(1)$, while the bosonic part of Killing supermetric is pseudo-Euclidean, $\eta_{\mu\nu} = -t_{\mu\rho}^\lambda t^\rho_{\nu\lambda} + C_{\mu\alpha}^\beta C_{\nu\beta}^\alpha = -Str(R_\mu^{ad} R_\nu^{ad}) = diag(1, -1, -1, -1)$. Of course, the latter by no means contradicts the compactness of $SU(2) \times U(1)$, since this symmetry, in its own right, does not fix the relative sign of η_{00} and η_{ij} . It is remarkable that the same indefinite signature of $\eta_{\mu\nu}$ is required for the existence of the matrix $\tilde{\Gamma}^\mu$ and hence for κ -invariance (no appropriate matrices $\tilde{\Gamma}^\mu$ exist with the choice $\eta_{\mu\nu} = \pm diag(1, 1, 1, 1)$).

Thus, the superstring sigma model associated with the coset

$$SU(2|1) \times SU(2|1)^* / U(2)_{diag}$$

also respects κ -supersymmetry. This model can probably be interpreted as describing a superstring moving in a special $D = 4, N = 2$ supergravity background with the isometry

supergroup $SU(2|1) \times SU(2|1)^*$ and $S^1 \times S^3$ (or $R^1 \times S^3$) as the bosonic submanifold⁷. Once again, counting physical bosonic and fermionic degrees of freedom indicates the potential presence of a world-sheet supersymmetry. Indeed, the number of physical bosonic coordinates equals $2(D - 2) = 4$ that coincides with the net number of fermionic coordinates which remain after utilizing κ -invariance ($\frac{1}{2}(4 + 4) = 4$).

4.9 Models based on direct sums of superalgebras

For any sum of the superalgebras considered above the identity (3.52) is obviously satisfied and the relevant Γ^μ and $\tilde{\Gamma}^\mu$ form reducible representations of Clifford algebra. As a result, the corresponding superstring sigma models are guaranteed to be κ -invariant. Here we describe a class of such models, with the bosonic manifolds being products of a flat Minkowski space and some nonabelian group manifolds. Many of these models reveal equal numbers of physical bosonic and fermionic coordinates and so have a chance to be world-sheet supersymmetric (and hence equivalent to some fermionic string models).

This particular class of the $G \times G^*/G_+$ sigma models is associated with the non-semisimple supergroups G generated by the following direct sums of superalgebras

$$\mathcal{G} = \mathcal{G}_G + \sum_1^L su(2|1) + \sum_1^K osp(2|1), \quad (4.11)$$

where \mathcal{G}_G is the Green algebra [12]. The flat Minkowski space part of the corresponding bosonic coordinates is related to the translation generators in \mathcal{G}_G ⁸ while the rest of these coordinates is valued in products of the $U(2)$ and $SL(2, R)$ group manifolds. The bosonic tangent space metric in the present case can be chosen in the form $\eta_{\mu\nu} = \text{diag}(1, -1, -1, \dots, 1, -1, -1, \dots, 1, -1, -1, \dots, -1, -1)$, first +1 coming from the flat Minkowski metric and the remaining $K + L$ ones from the $SL(2, R)$ and $U(1)$ factors.

Let us deduce the conditions under which these models possess equal numbers of physical bosonic and fermionic coordinates. Denoting by D the full dimension of the bosonic manifold, we have

$$D = d + 3K + 4L, \quad (4.12)$$

d being the dimension of Minkowski space. Thus, in the light-cone gauge one is left with

$$N_b = 2(D - 2) = 2(d + 3K + 4L - 2) \quad (4.13)$$

essential bosonic degrees of freedom ($D - 2$ left and $D - 2$ right movers). Now recall that Green's superalgebras exist only in dimensions $d = 3, 4, 6, 10$ and have, respectively, 4, 8, 16 and 32 real fermionic generators [12] (two Majorana spinors for $d = 3, 4$, two complex Weyl spinors for $d = 6$ and two Majorana-Weyl spinors for $d = 10$). Further,

⁷The $U(1)$ generator included in the coset space is realized as shifts of the coordinate $x^0(\xi)$, so we may treat the abelian factor in the bosonic submanifold to be compact S^1 or noncompact R^1 depending on the boundary conditions imposed on x^0 .

⁸To avoid a misunderstanding, we point out that the Minkowski space Lorentz group is assumed to be placed in the stability subgroup.

as was mentioned in Sec.3.6, local κ -supersymmetry is capable to gauge away half of the fermionic coordinates associated with one of two spinor generators of the Green superalgebra, leaving intact the coordinates related to the second generator. Then it is easy to obtain that the Green supergroup sector of our $G \times G^*/G_+$ sigma model contributes

$$N_G = 2(3d - 6) \quad (4.14)$$

of physical fermionic degrees of freedom. The total number of such degrees equals

$$N_f = 2(3d - 6 + K + 2L). \quad (4.14)$$

Finally, the condition

$$N_b = N_f$$

yields the simple necessary criterion for the existence of world-sheet supersymmetry in the considered class of models

$$d - K - L = 2. \quad (4.15)$$

Substituting the admissible values 3, 4, 6, and 10 of d into (4.15), we may figure out those values of K, L and D (eq.(4.12))⁹ which correspond to the potentially world-sheet supersymmetric superstring models

The number of possible solutions rapidly increases with growing of d , so we limit ourselves to presenting several noticeable examples.

First, we quote the solutions corresponding to four-dimensional Minkowski space

$$\underline{d = 4}$$

$$1) K = 0, L = 2 (D = 12); \quad 2) K = 1, L = 1 (D = 11); \quad (4.16)$$

$$3) K = 2, L = 0 (D = 10);$$

Third solution in (4.16) seems to be most interesting because the related sigma model is expected to represent the superstring moving in a special $D = 10$, $N = 2$ supergravity background which is reduced in the bosonic sector to the product of four-dimensional Minkowski space and six-dimensional non-compact group manifold $\sim SO(1, 2) \times SO(1, 2)$.

Other interesting models arise at the choices

$$d = 3, K = 1, L = 0 (D = 6) \quad (4.17)$$

and

$$d = 3, K = 0, L = 1 (D = 7) \quad (4.18)$$

The model (4.17) can be interpreted as the superstring in a $D = 6$, $N = 2$ supergravity background the bosonic part of which is the product of three-dimensional Minkowski space

⁹Note that the whole dimension of the bosonic subspace, as it follows from (4.12), is summed up from the “magic” dimensions 3, 4, 6, 10 needed for the existence of the closed Wess-Zumino three-forms having non-vanishing flat superspace limit and for κ -supersymmetry.

and three-dimensional non-compact group manifold $\sim SO(1, 2)$. The model characterized by the parameters (4.18) represents a $D = 7$, $N = 2$ superstring moving in the background which is reduced in the bosonic sector to the product of three-dimensional Minkowski space and four-dimensional compact manifold $S^1 \times S^3$ related to the group $U(1) \times SU(2)$ (the circle S^1 can be replaced by a real line R^1 , see last footnote).

We hope to return to studying these models elsewhere.

5 CONCLUSION

In this paper we have constructed the new WZW models on supergroups which can be interpreted as models of $N = 2$ GS superstrings moving in supergroup spaces. In doing so, we exploited the basic concepts of our reformulation of ordinary $N = 2$ GS superstring as a WZW sigma model on the product of two $N=1$ supertranslation groups [9, 10]. It was not clear before how to consistently construct nonabelian versions of the GS $N = 2$ superstring proceeding from its conventional sigma model interpretation [7, 8] or by choosing an appropriate supergravity background in the model of a superstring moving in the general curved $N=2$ superspace [19]. So, the main merit of our formulation of ordinary $N = 2$ GS superstring has to be seen in the possibility to straightforwardly extend it to the nonabelian case.

The major idea of our approach is to consider a WZW sigma model with the target coset space $G_1 \otimes G_2 / G_+$ where $G_1 \otimes G_2$ is a direct product of two supergroups dual to each other in Cartan's sense and G_+ is the maximal even diagonal subgroup of $G_1 \otimes G_2$. We have shown that if G_1 and G_2 have a flat limit coinciding with two $N=1$ supertranslation groups, then our model in this limit coincides with the $N=2$ GS superstring theory. We have deduced the equations of motion and proved their classical integrability (for arbitrary gauge fixing). This is important for setting up a self-consistent quantum version of our models which in general possess no κ -supersymmetry and for this reason may be not conformally invariant.

We think that our construction can be extended to the superstring theories having nonequivalent left and right sectors in the space of string variables (similarly to the heterotic string). Besides, we expect that it can be directly applied to the case when G_1 and G_2 are arbitrary Z_2 graded groups (not necessarily supergroups). Proceeding in this way, it seems possible to construct new integrable sigma models and to formulate new string theories, perhaps, with the critical dimensions close to the physical value of four. One more possible application of our models is an unconventional way of introducing gauge degrees of freedom into string theories without using Chan-Paton factors or additional fermions [20].

The important problems concerning the Hamiltonian formulation and quantization of the present models as well as their connection with $N = 2$ superstring moving in an arbitrary supergravity background will be addressed in our forthcoming paper.

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